数理解析研究所講究録
756巻 1991年 137-150

Quotients of hypersurface sections of toric singularities.

東北学院大学 土橋宏康 (Hiroyasu TSUCHIHASHI)

Introduction. In the previous paper [4], we studied hypersurface sections (X,y) of toric singularities (Y,y). In this paper, we continue the study of (X,y) and also study the quotients of (X,y) by groups G of toric actions, which are finite subgroups of the algebraic torus. Let $Y' = Y/G$, let $X' = X/G$ and let y' be the images of y under the quotient maps $Y \to Y'$. Then (Y',y') are also toric singularities, while (X',y') may not be hypersurface sections of (Y',y'). Hence more singularities appear as (X',y') than those as (X,y). For instance, the multiplicities of simple elliptic singularities which are hypersurface sections of toric singularities are smaller than 10 (see [4]), while there are no restrictions on those of their quotients by toric actions. We show that also for those quotients (X',y') explicit resolutions are obtained by the Newton boundaries of the defining equations f of X, if f are non-degenerate. Moreover, if (Y,y) are Gorenstein, then we can calculate the plurigenera of (X',y').

In §1, we recall some facts about toric singularities and their hypersurface sections and give a necessary condition for

(X,y) to be isolated.

In §2, we give a necessary and sufficient condition for finite groups G of toric actions mapping X on themselves to have no fixed points on hypersurface sections (X,y) of toric singularities, and that for the quotients (X',y') of (X,y) by G to be Gorenstein singularities.

In §3, we construct resolutions of (X',y'), using torus embeddings and as an application, we give a sufficient condition for (X',y') to be isolated singularities.

In §4, under the assumption that (X,y) are Gorenstein, we give a formula of calculating plurigenera of (X',y') via shapes of Newton boundaries and a necessary and sufficient condition for (X',y') to be purely elliptic of $(0,\dim X'-1)$ type singularities (see [2], for the definition of purely elliptic of $(0,i)$ type singularities).

In §5, we give two examples.

§1 Toric singularities and their hypersurface sections.

We use the notations and the terminologies in [3]. Let $N = \mathbb{Z}^{n+1}$ be a free $\mathbb{Z}$-module of rank $n + 1 \geqq 3$ and let M be its dual module with canonical pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$. Let σ be an $(n+1)$-dimensional strongly convex rational cone in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, i.e., σ is generated by finite elements in N containing linearly independent $(n+1)$ elements and $\sigma \cap (-\sigma)$

$= \{0\}$. Let $Y = \mathrm{Spec}\mathbb{C}[\sigma^* \cap M]$ and let $z^v : Y \to \mathbb{C}$ be the character of v, which is the natural extension of $v \otimes 1_{\mathbb{C}^\times}$: $\mathrm{Spec}\mathbb{C}[M] \simeq (\mathbb{C}^\times)^{n+1} \to \mathbb{C}^\times$ for each v in $\sigma^* \cap M$. Recall that $\mathrm{orb}(\tau) = \{ x \in \mathrm{Spec}\mathbb{C}[\tau^* \cap M] \mid z^v(x) = 0$ for all $v \in (\tau^* \setminus \tau^\perp) \cap M \}$ for each face τ of σ, where $\tau^\perp = \{ v \in M_{\mathbb{R}} \mid \langle v,u \rangle = 0$ for all $u \in \tau \}$. We easily obtain:

Lemma 1.1. Let τ be a face of σ and let v be an element in $\sigma^* \cap M$. Then the following three conditions are equivalent.

1) $z^v(x) \neq 0$ for a point x in $\mathrm{orb}(\tau)$.

2) $z^v(x) \neq 0$ for all points x in $\mathrm{orb}(\tau)$.

3) v is in $\tau^\perp$.

The set $\mathrm{orb}(\sigma) = \{ x \in Y \mid z^v(x) = 0$ for all $v \in (\sigma^* \cap M) \setminus \{0\} \}$ consists of only one point, which we denote by y. Let $X = \{f = 0\}$ be the hypersurface section of Y defined by an element $f = \sum_{v \in \sigma^* \cap M} c_v z^v$ in $\mathbb{C}[\sigma^* \cap M]$. Throughout this paper, we assume that X contains y, i.e., $c_0 = 0$. Let $\mathrm{Supp}(f) = \{ v \in \sigma^* \cap M \mid c_v \neq 0 \}$. For each face τ of σ, we can take a basis $\{v_1, v_2, \ldots, v_{n+1}\}$ of M so that $\{v_1, v_2, \ldots, v_\ell\}$ is a basis of $\tau^\perp \cap M$, where $\ell = \dim \tau^\perp = n + 1 - \dim \tau$. Let $z_i = z^{v_i}$ for $i = 1$ through ℓ and let $\Phi(x) = (z_1(x), z_2(x), \ldots, z_\ell(x))$ for x in $\mathrm{orb}(\tau)$. Then the map $\Phi : \mathrm{orb}(\tau) \to (\mathbb{C}^\times)^\ell$ is an isomorphism and

$(z^v \cdot \Phi^{-1})(t_1, t_2, \ldots, t_\ell) = t_1^{a_1} t_2^{a_2} \ldots t_\ell^{a_\ell}$, if $v = a_1 v_1 + a_2 v_2 + \ldots + a_\ell v_\ell \in \tau^\perp \cap M$. Hence by Lemma 1.1, we obtain:

Lemma 1.2. $X \supset \mathrm{orb}(\tau)$ if and only if $\mathrm{Supp}(f) \cap \tau^\perp = \emptyset$. $X \cap \mathrm{orb}(\tau) = \emptyset$ if and only if $\mathrm{Supp}(f) \cap \tau^\perp$ consists of only one point.

If $\mathrm{Supp}(f) \cap \tau^\perp = \emptyset$ for a 1-dimensional face τ of σ, then by the above lemma, X contains the codimension 1 subvariety $\overline{\mathrm{orb}(\tau)}$ of Y. On the other hand, $X \cap T_N \neq \emptyset$, if and only if $\#\mathrm{Supp}(f) > 1$, where $T_N := \mathrm{Spec}\mathbb{C}[M] = Y \setminus \cup_\tau$ are 1-dimensional faces of σ $\overline{\mathrm{orb}(\tau)}$. Hence if X is irreducible and $\#\mathrm{Supp}(f) > 1$, then $\mathrm{Supp}(f) \cap \mu^* \neq \emptyset$ for all n-dimensional faces μ^* of σ^*. In the next section, beside N we take a free $\mathbb{Z}$-module N' with $N'_{\mathbb{R}} = N_{\mathbb{R}}$. Hence we need the following definition.

Definition 1.3. A rational polyhedral cone τ in $N_{\mathbb{R}}$ is said to be non-singular with respect to N, if τ is generated by a part of a basis of N.

Proposition 1.4. Assume that (X,y) is isolated. Then any (n-1)-dimensional face τ of σ is non-singular with respect to N and if an n-dimensional face τ of σ is not non-singular with respect to N, then $\mathrm{Supp}(f) \cap \tau^\perp \neq \emptyset$.

§2 Quotients of (X,y) by finite groups of toric actions.

Let M' be a submodule of M with finite index and let $N' = \{ u \in N_{\mathbb{Q}} \mid \langle v,u\rangle \in \mathbb{Z}$ for all $u \in M' \}$. Let $Y' = \mathrm{Spec}\mathbb{C}[\sigma^* \cap M']$. Then Y' is the quotient of Y under the group $G := N'/N$ and the set $\{ x \in Y' \mid z^v(x) = 0$ for all $v \in (\sigma^* \cap M') \setminus \{0\} \}$ consists of only one point y', which is the image of y under the quotient map $Y \to Y'$. Here the group G acts on Y in the following way: $g^*z^v = e^{2\pi\sqrt{-1}\langle v,u\rangle}z^v$ for each element $g = [u]$ in G. Hence $g^*z^v = z^v$ for all g in G, if and only if v is in M'. Assume that there exists an element v_0 in M with $\mathrm{Supp}(f) \subset v_0 + M'$. Then for each element $g = [u]$ in G, $g^*f = e^{2\pi\sqrt{-1}\langle v_0,u\rangle}f$. Hence X is invariant under the action of G. Let $X' = X/G$. Then X' is the codimension 1 subvariety of Y' with the ideal $(f\cdot\mathbb{C}[\sigma^*\cap M]) \cap \mathbb{C}[\sigma^*\cap M']$, which is generated by $\{ z^vf \mid v \in \sigma^*\cap M,\ v_0 + v \in M' \}$.

Lemma 2.1. Let τ be a face of σ. If $N \cap \mathbb{R}\tau \neq N' \cap \mathbb{R}\tau$ (resp. $N \cap \mathbb{R}\tau = N' \cap \mathbb{R}\tau$), then all points (resp. no points) of $\mathrm{orb}(\tau)$ are fixed points of G.

Proposition 2.2. G has no fixed points on $X \setminus \{y\}$ (resp. $U \setminus \{y\}$ for an open neighborhood U of y in X), if and only if the following two conditions are satisfied.

1) For each $(n-1)$-dimensional face τ of σ, $N \cap \mathbb{R}\tau =$

$N' \cap \mathbb{R}\tau$.

2) For each n-dimensional face τ of σ, if $N \cap \mathbb{R}\tau \neq N' \cap \mathbb{R}\tau$, then $\mathrm{Supp}(f) \cap \tau^{\perp}$ consists of only one point (resp. is not empty).

Let $N'' = \{ u \in N' \mid \langle v_0, u\rangle \in \mathbb{Z} \}$ and let $G'' = \{ g \in G \mid g^*f = f \}$. Then $G'' = N''/N$ and the quotient X/G'' of X by G'' is a hypersurface section of that Y/G'' of Y, which is also a toric singularity. Moreover, X' is the quotient of X/G'', by the group $G/G'' = N'/N''$ of toric actions. Hence we may only consider the case that $G'' = \{id\}$.

Proposition 2.3. Assume that $N \cap \mathbb{R}\tau \neq N' \cap \mathbb{R}\tau$ for a 1-dimensional face τ of σ. If f is not in $\mathbb{C}[\sigma^* \cap M'']$, then $X \supset \overline{\mathrm{orb}(\tau)}$, where $M'' = \{ v \in M \mid \langle v,u\rangle \in \mathbb{Z}$ for all $u \in N + (N' \cap \mathbb{R}\tau) \}$.

Let τ be a 1-dimensional face of σ. If $X \supset \overline{\mathrm{orb}(\tau)}$, then X is not irreducible or $X = \overline{\mathrm{orb}(\tau)}$ is a toric variety. On the other hand, if f is in $\mathbb{C}[\sigma^* \cap M'']$, then X' is the quotient of the hypersurface section $\{f = 0\}$ of $\mathrm{Spec}\,\mathbb{C}[\sigma^* \cap M'']$ by the group $N'/N+(N' \cap \mathbb{R}\tau)$. Hence by the above proposition, we may only consider the case that $N \cap \mathbb{R}\tau = N' \cap \mathbb{R}\tau$ for all 1-dimensional faces τ of σ. Namely, the codimension of the branch locus of the quotient map $q : Y \to$

Y' is greater than 1, by Lemma 2.1. In this case, if a face μ of σ with $\dim \mu > 1$ is non-singular with respect to N', then μ is non-singular also with respect to N and $N \cap \mathbb{R}\mu = N' \cap \mathbb{R}\mu$, i.e., q does not ramify along $\mathrm{orb}(\mu)$.

Proposition 2.4. Assume that $\{ g \in G \mid g^*f = f \} = \{\mathrm{id}\}$, that (X',y') is isolated and that $N \cap \mathbb{R}\tau = N' \cap \mathbb{R}\tau$ for each 1-dimensional face τ of σ. Then G is a cyclic group.

Proposition 2.5. Assume that $N \cap \mathbb{R}\tau = N' \cap \mathbb{R}\tau$ for each 1-dimensional face τ of σ, that $\dim \mathrm{Sing}(X') \leq n - 2$ and that (Y,y) is Gorenstein, i.e., there exists the element $v(\sigma)$ in M such that $\langle v(\sigma), u_\tau \rangle = 1$ for each 1-dimensional face τ of σ, where u_τ is the generator of $N \cap \tau$ (see [4, Proposition 1.2]). Then (X',y') is r-Gorenstein (, i.e., there exists a nowhere vanishing holomorphic r-ple n-form on $U' \setminus \mathrm{Sing}(U')$ for an open neighborhood U' of y' in X'), if and only if $\mathrm{Supp}(f^r) \subset rv(\sigma) + M'$.

§3 Resolutions of (X',y').

Throughout the rest of this paper, we assume that for all 1-dimensional faces τ of σ, $\mathrm{Supp}(f) \cap \tau^\perp \neq \emptyset$, i.e., X' does not contain $\mathrm{orb}(\tau)$, by Lemma 1.2. Recall that the Newton polyhedron $\Gamma_+(f)$ of $f = \sum c_v z^v \in \mathbb{C}[\sigma^* \cap M]$ is the convex hull of $\cup_{v\in \mathrm{Supp}(f)}(v+\sigma^*)$ and that the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$. Let $d(u) = \min\{ \langle v,u\rangle \mid v \in \Gamma_+(f) \}$ and let $\Delta(u) = \{ v \in \Gamma_+(f) \mid \langle v,u\rangle = d(u) \}$ for each point u in σ. For a face $\Delta = \Delta(u_0)$ ($u_0 \in \sigma$) of $\Gamma_+(f)$, let $\Delta^* = \{ u \in \sigma \mid \Delta(u) \supset \Delta \}$. Recall that the dual Newton decomposition $\Gamma^*(f)$ of f is $\{ \Delta^* \mid \Delta$ are faces of $\Gamma_+(f) \}$, which is an r.p.p. decomposition of N' as well as of N with $|\Gamma^*(f)| = \sigma$. We say that f is non-degenerate, if

$$\partial f_\Delta/\partial z_1 = \dots = \partial f_\Delta/\partial z_{n+1} = 0$$

has no solutions in $T := \mathrm{Spec}\mathbb{C}[M] \subset Y$ for each face Δ of $\Gamma(f)$, where $f_\Delta = \Sigma_{v\in\Delta\cap \mathrm{Supp}(f)} c_v z^v$ and $(z_1,z_2, \dots ,z_{n+1})$ is a global coordinate of T, for instance, $z_i = z^{v_i}$ for a basis $\{v_1,v_2, \dots ,v_{n+1}\}$ of M.

Proposition 3.1. If f is non-degenerate, then X' is irreducible at y'.

Note that if (X',y') is isolated and f is non-degenerate, then (X',y') is normal, by the above proposition.

Proposition 3.2. There exists a subdivision Σ of $\Gamma^*(f)$ consisting of non-singular cones with respect to N' such that if a cone τ in $\Gamma^*(f)$ is non-singular with respect to N', then τ is also in Σ.

Take a subdivision Σ of $\Gamma^*(f)$ satisfying the condition of the above proposition. Let $\hat{Y}' = T_{N'}\mathrm{emb}(\Sigma)$ and let $\tilde{X}'$ be the proper transformation of X' under the map $\Pi' : \hat{Y}' \to Y'$ induced by the natural morphism $(N',\Sigma) \to (N',\{\text{faces of } \sigma\})$ of r.p.p. decompositions. Then $\tilde{X}'$ is the closure of $X' \cap T_{N'}$ in $\hat{Y}'$, by the first assumption in this section.

Proposition 3.3. Assume that f is non-degenerate. Then there exists an open neighborhood U' of y' in Y' such that $(\Pi')^{-1}(U') \cap \tilde{X}'$ is non-singular.

Proposition 3.4. Assume that f is non-degenerate, that each $(n-1)$-dimensional face τ of σ is a non-singular cone with respect to N' and that $\mathrm{Supp}(f) \cap \mu^* \neq \emptyset$ for each 1-dimensional face μ^* of σ^*. Then the restriction of Π' to $(\Pi')^{-1}(U' \setminus \{y'\})$ is an isomorphism for an open neighborhood U' of y' in X'. Hence (X',y') is an isolated singularity.

§4 Plurigenera.

Throughout this section, we assume that (Y,y) is Gorenstein, i.e., there exists the point $v(\sigma)$ in M such that $\langle v(\sigma), u_\tau\rangle = 1$ for all 1-dimensional faces τ of σ, where u_τ are the generators of $N \cap \tau$. Moreover, we assume that G has no fixed points on $U \setminus \{y\}$ for an open neighborhood U of y in X (see Proposition 2.2). We keep the notations in the previous section. Let $E'(\tau) = \hat{X}'\cdot\overline{orb(\tau)}$ for each cone τ in $\Sigma_1 := \{\tau \in \Sigma \mid \dim\tau = 1, \mathrm{Int}(\tau) \subset \mathrm{Int}(\sigma)\}$. Then $E' := (\pi')^{-1}(y') = \sum_{\tau\in\Sigma_1} E'(\tau)$ and $E'(\tau) \neq \emptyset$ if and only if $\dim \Delta(u_\tau) \geq 1$, where u_τ is a generator of τ and $\pi' = (\Pi')_{|\hat{X}'}$. Let $\omega = \mathrm{Res}(z^{v(\sigma)}((dz_1/z_1)\wedge \ldots \wedge(dz_{n+1}/z_{n+1})/f)$, where $(z_1, z_2, \ldots, z_{n+1})$ is a global coordinate of T_N. Recall that there exists an element v_0 in M with $\mathrm{Supp}(f) \subset v_0 + M'$. If $v \in \sigma^*$ and $v + mv(\sigma) \in mv_0 + M'$, then the holomorphic m-ple n-form $z^v\omega^m$ is G-invariant. Since the quotient map $U \setminus \{y\} \to U' \setminus \{y'\}$ is unramified, there exists the holomorphic m-ple n-form on $X' \setminus \{y'\}$ whose pull back is equal to $z^v\omega^m$, where $U' = \pi'(U)$. We denote it by $q_*z^v\omega^m$.

Lemma 4.1. Let τ be in Σ_1. If $v \in \sigma^*$ and $v + mv(\sigma) \in mv_0 + M'$, then $(\pi')^*(q_*z^v\omega^m)$ has zeros along $E'(\tau)$ of order $\langle v, u_\tau\rangle + m\langle v(\sigma), u_\tau\rangle - m - md(u_\tau)$, where u_τ is the generator of $N' \cap \tau$.

Here we note that $\langle v, u_\tau\rangle + m\langle v(\sigma), u_\tau\rangle > md(u_\tau)$ for all τ in Σ_1 with $E'(\tau) \neq \emptyset$, if and only if $v + mv(\sigma) \in$

$m\mathrm{Int}(\Gamma_+(f))$. Hence as a generalization of [5, Theorem 2.2], we obtain the following theorem, by the subsequent two propositions.

Theorem 4.2. Assume that f is non-degenerate and that (X',y') is isolated. Then

$$\delta_m(X',y') = \#\{\ v \in \sigma^* \cap M \mid v + mv(\sigma) \in (mv_0 + M') \cap m\Gamma_-(f)\ \}$$
$$- \#\{\ v \in \sigma^* \cap M \mid v + mv(\sigma) \in ((m-1)v_0 + M') \cap (m-1)\Gamma_-(f)\ \},$$

where $\Gamma_-(f) = \sigma \setminus \mathrm{Int}(\Gamma_+(f))$.

Proposition 4.3. Let $\pi : (\hat{X},E) \to (X,y)$ be a good resolution of (X,y). If $\pi^*(h_{|X\cap U}\cdot\omega^m) \in H^0(\hat{X},\mathcal{O}(mK+(m-1)E))$ for a holomorphic function h on a neighborhood U of y in Y, then there exists a holomorphic function h' on U such that $h'_{|X\cap U} = h_{|X\cap U}$ and that $mv(\sigma) + \mathrm{Supp}(h') \subset m\mathrm{Int}(\Gamma_+(f))$.

Proposition 4.4. Let U be a neighborhood of y in Y which is invariant under the action of G and let h be a holomorphic function on U. Assume that there exists an element v' in M such that $(g_{|X\cap U})^*h_{|X\cap U} = e^{2\pi\sqrt{-1}\langle v',u\rangle}h_{|X\cap U}$ for each $g = [u]$ in G. Then there exists a holomorphic function h' on U such that $h'_{|X\cap U} = h_{|X\cap U}$, that $\mathrm{Supp}(h') \subset \mathrm{Supp}(h)$ and that $g^*h' = e^{2\pi\sqrt{-1}\langle v',u\rangle}h'$ for each $g = [u]$ in G.

Corollary 4.5. Assume that f is non-degenerate and that (X',y') is isolated. Then (X',y') is purely elliptic, i.e., $\delta_m(X',y') = 1$ for all positive integers m, if and only if $\mathrm{Supp}(f) \subset v(\sigma) + M'$ and $v(\sigma) \in \Gamma(f)$. Moreover, under the above condition, (X',y') is purely elliptic of $(0,i)$-type for $i = 1$ through $n - 1$ (resp. $(0,0)$-type), if and only if $\dim \Delta = i + 1$ (resp. 0 or 1), where Δ is the face of $\Gamma(f)$ with $v(\sigma) \in \mathrm{Int}(\Delta)$.

§5 Examples.

1. Let $n = 2$, let $\{u_1,u_2,u_3\}$ be a basis of N and let $\{v_1,v_2,v_3\}$ be the basis of M dual to $\{u_1,u_2,u_3\}$. Let $\sigma = \mathbb{R}_{\geq 0}u_1 + \mathbb{R}_{\geq 0}u_2 + \mathbb{R}_{\geq 0}u_3$, let $M' = \{ v \in M \mid \langle v, \frac{1}{5}(u_1+u_2+u_3)\rangle \in \mathbb{Z} \}$ and let $f = z_1z_2z_3 + z_1^8 + z_2^8 + z_3^8$, where $z_i = z^{v_i}$ for $i = 1, 2, 3$. Then f is non-degenerate and $\mathrm{Supp}(f) \subset (v_1+v_2+v_3) + M'$. Since $v(\sigma) = v_1 + v_2 + v_3$ is a vertex of $\Gamma(f)$, (X',y') is purely elliptic of $(0,0)$ type, i.e., (X',y') is a cusp singularity. Let Σ be the set of the faces of the cones $\mathbb{R}_{\geq 0}u_i + \mathbb{R}_{\geq 0}u_{i+1} + \mathbb{R}_{\geq 0}t_{i+1}$, $\mathbb{R}_{\geq 0}u_i + \mathbb{R}_{\geq 0}t_i + \mathbb{R}_{\geq 0}t_{i+1}$ and $\mathbb{R}_{\geq 0}t_i + \mathbb{R}_{\geq 0}t_{i+1} + \mathbb{R}_{\geq 0}s$ for $i = 1, 2, 3$, where $u_4 = u_1$, $t_4 = t_1$, $s = u_1 + u_2 + u_3$ and $t_i = s + 5u_i$. Then Σ is a subdivision of $\Gamma^*(f)$ satisfying the condition of Proposition 3.2. Let $E_i = E'(\mathbb{R}_{\geq 0}t_i)$ and let $E_0 = E'(\mathbb{R}_{\geq 0}s)$. Then we easily see that $E_0 = \emptyset$, that E_i are rational curves with E_i^2

$= -7$ and that $E_1E_2 = E_2E_3 = E_3E_1 = 1$.

2. Assume that n is an even number. Let $\{e_1, e_2, \ldots, e_{n+1}\}$ be a basis of $N = \mathbb{Z}^{n+1}$ and let $\{d_1, d_2, \ldots, d_{n+1}\}$ be the basis of M dual to $\{e_1, e_2, \ldots, e_{n+1}\}$. Let $\sigma = \mathbb{R}_{\geqq 0}u_1 + \mathbb{R}_{\geqq 0}u_2 + \cdots + \mathbb{R}_{\geqq 0}u_{n+1}$, where $u_i = e_i + e_{n+1}$ for $i = 1$ through n and $u_{n+1} = -e_1 - e_2 - \cdots - e_n + e_{n+1}$. Then each n-dimensional face of σ is non-singular with respect to N and σ^* is generated by v_1, v_2, ... and v_{n+1}, where $v_{n+1} = -d_1 - d_2 - \cdots - d_n + d_{n+1}$ and $v_i = (n+1)d_i + v_{n+1}$ for $i = 1$ through n. Let $N' = N + \frac{1}{n+1}(e_1 + e_2 + \cdots + e_{n+1})\mathbb{Z}$ and let $M' = \{ v \in M \mid \langle v,u \rangle \in \mathbb{Z}$ for all $u \in N' \}$. Then $N \cap \mathbb{R}\tau = N' \cap \mathbb{R}\tau$ for all n-dimensional faces τ of σ. Let $f = z^{v_1} + z^{v_2} + \cdots + z^{v_{n+1}}$. Then f is non-degenerate, $\mathrm{Supp}(f) \subset v_1 + M'$ and $v(\sigma) = d_{n+1} \in \Gamma(f)$. Hence (X', y') is isolated, by Proposition 3.4. Since $k(v_1 - v(\sigma)) \notin M'$ for $k = 1$ through n and $(n+1)(v_1 - v(\sigma)) \in M'$, we have $\delta_m(X', y') = 1$ or 0, accordingly m is a multiple of $n + 1$ or not, by Theorem 4.2.

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